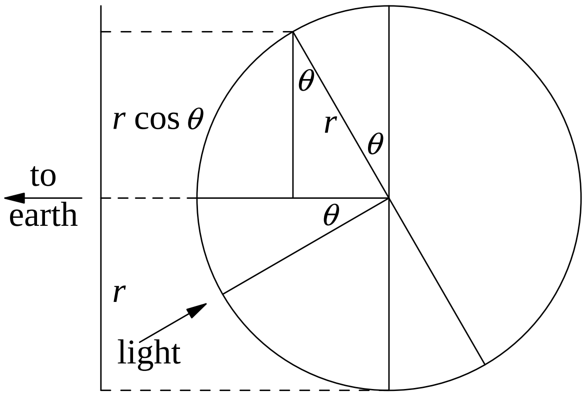
1. Find planet phase based on sun angle

The goal in this section is to determine, for any planet, what fraction of the the planet as viewed from earth is illuminated by the sun. So a phase of "new" would be 0, and a phase of "full" would be 1.

Set the following values:
Angle between the earth and sun, as viewed from the planet: θ
Radius of a cross section of the planet: r



The diagram shows any cross section of the planet parallel to earth's orbit. Of the side seen from earth (the left side), the bottom half is fully illuminated, width r. In the top half, a width of $r \cos\theta$ is illuminated. So the total fraction illuminated is

$$\frac{r + r\cos\theta}{2r} = \frac{1 + \cos\theta}{2}$$

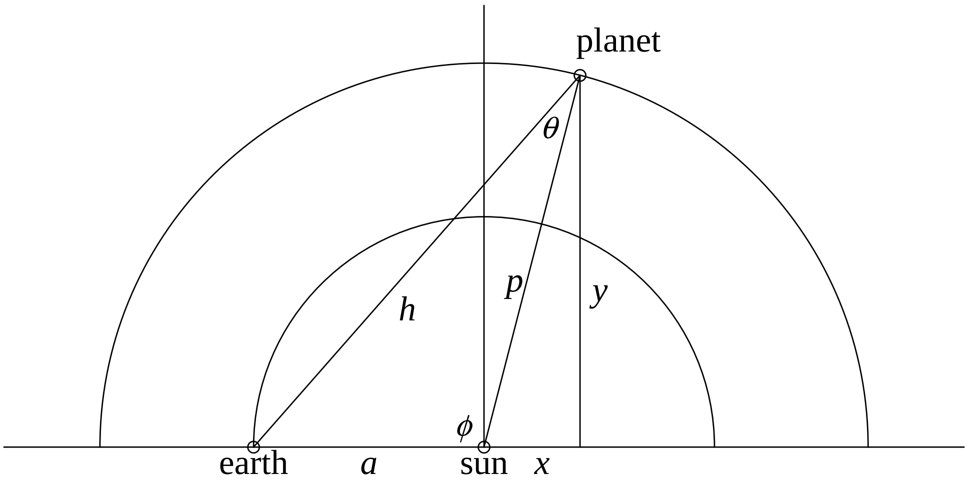
If θ is more than 90 degrees, a similar diagram gives the same result.

2. Find θ based on orbital positions of a superior planet

First we will deal with superior planets.

In this section, given the positions of earth and the planet in their orbits, we find θ, the angle between the earth and sun, as viewed from the planet. We assume that the orbits are circular and coplanar.

Set the following values:
Radius of earth's orbit: a
Radius of planet's orbit: p
Sun's position: $(0, 0)$
Earth's position: $(-a, 0)$
Planet's position: $(x, y) = (x, \sqrt{p^2 - x^2})$
Distance from earth to planet: h
Angle from earth to planet, as seen from sun: ϕ
Angle from earth to sun, as seen from planet: θ



Using the Pythagorean theorem:

$$h^2 = (a+x)^2 + y^2 = (a+x)^2 + (p^2 - x^2) = a^2 + 2ax + p^2$$

By the law of cosines:

$$h^2 = a^2 + p^2 - 2ap\cos\phi$$

Solve that for $\cos\phi$ and plug in the value for h^2:

$$\cos\phi = \frac{a^2+p^2-h^2}{2ap} = \frac{a^2+p^2-(a^2+2ax+p^2)}{2ap} = \frac{-2ax}{2ap} = \frac{-x}{p}$$

From $\cos\phi$ find $\sin\phi$:

$$\sin\phi = \pm\sqrt{1-\cos^2\phi} = \pm\sqrt{1-x^2/p^2}$$

ϕ is between 0 and 180 degrees, so use the positive square root:

$$\sin\phi = \sqrt{1-x^2/p^2}$$

By the law of sines:

$$\frac{h}{\sin\phi} = \frac{a}{\sin\theta}$$

Solve for $\sin\theta$, plugging in values for h and $\sin\phi$:

$$\sin\theta = \frac{a\sin\phi}{h} = \frac{a\sqrt{1-x^2/p^2}}{\sqrt{a^2+p^2-2ap\cos\phi}} = \frac{a\sqrt{1-x^2/p^2}}{\sqrt{a^2+2ax+p^2}} = a\sqrt{\frac{1-x^2/p^2}{a^2+2ax+p^2}}$$

Thales' theorem says that an angle inscribed in a semicircle is a right angle, like the angle from $(-p,0)$ to (x,y) to $(p,0)$. Moving $(-p,0)$ to $(-a,0)$, and $(p,0)$ to $(x,0)$, the angle clearly gets smaller; so θ is less than a right angle. And so we can say:

$$\theta = \text{Arcsin}\left[a \sqrt{\frac{1 - x^2/p^2}{a^2 + 2ax + p^2}} \right]$$

This is θ as a function of x.

3. Find where the maximum value of θ occurs.

When $x = -p$ (opposition) or $x = p$ (conjunction), we know that $\theta = 0$. In between those points, it is greater than 0. The goal of this section is to find what value of x results in the greatest value for θ.

To find the maximum (and minimum) values of this function, take the derivative and set it to 0. The derivative is:

$$\frac{d\theta}{dx} = \frac{d}{dx}\left\{ \text{Arcsin}\left[a \sqrt{\frac{1 - x^2/p^2}{a^2 + 2ax + p^2}} \right] \right\}$$

$$= \frac{1}{\sqrt{1 - \dfrac{a^2(1 - x^2/p^2)}{a^2 + 2ax + p^2}}} \cdot \frac{(a^2 + 2ax + p^2)(-(2a^2/p^2)x) - a^2(1 - x^2/p^2)(2a)}{(a^2 + 2ax + p^2)^2}$$

Set to 0 and multiply each side by the denominators:

$$0 = -2\,\frac{a^4}{p^2}\,x - 4\,\frac{a^3}{p^2}\,x^2 - 2a^2 x - 2a^3 + 2\,\frac{a^3}{p^2}\,x^2$$

$$= -2a^2 x - 2ax^2 - p^2 x - ap^2 + ax^2 = ax^2 + (a^2 + p^2)x + ap^2$$

Using the quadratic formula:

$$x = \frac{-(a^2 + p^2) \pm \sqrt{(a^2 + p^2)^2 - 4a^2 p^2}}{2a} = \frac{-(a^2 + p^2) \pm \sqrt{a^4 + 2a^2 p^2 + p^4 - 4a^2 p^2}}{2a}$$

$$= \frac{-(a^2 + p^2) \pm \sqrt{(a^2 - p^2)^2}}{2a} = \frac{-(a^2 + p^2) \pm (a^2 - p^2)}{2a}$$

So

$$x = \frac{-2p^2}{2a} = \frac{p^2}{a} \quad \text{or} \quad x = \frac{-2a^2}{2a} = -a$$

p^2/a is greater than p, and x cannot be greater than p. So $x = -a$, which is the point of quadriture.

4. Find maximum value of θ and the phase at that time

Now we find the maximum value of θ. That will result in the minimum phase, because as θ increases from 0 to 180 degrees, $\cos\theta$ gets smaller, causing the phase, $(1 + \cos\theta)/2$, to get smaller.

We plug the above $-a$ value for x into the equation for θ.

$$\theta = \text{Arcsin}\left[a\sqrt{\frac{1 - a^2/p^2}{a^2 - 2a^2 + p^2}}\right] = \text{Arcsin}\left[a\sqrt{\frac{1 - a^2/p^2}{p^2 - a^2}}\right] = \text{Arcsin}\left[\frac{a}{p}\sqrt{\frac{p^2 - a^2}{p^2 - a^2}}\right]$$

So

$$\theta = \text{Arcsin}\,\frac{a}{p}$$

using the positive square root, since θ is nonnegative.

Using this, we can now find the phase (the fraction of the side facing earth that is illuminated). Based on the previous equation:

$$\sin\theta = \frac{a}{p}$$

So

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - a^2/p^2}$$

The phase is

$$\frac{1 + \cos\theta}{2} = \frac{1 + \sqrt{1 - a^2/p^2}}{2}$$

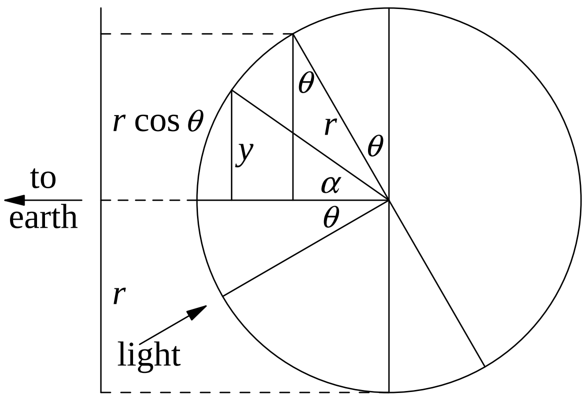
Note that if p is very close to a (a planet just barely farther from the sun than earth), the minimum phase approaches 1/2. If p is very large, the minimum phase approaches 1.

5. Find brightness based on θ

Brightness depends on the phase and on the distance from the planet to earth. It also depends on the albedo, on whether the surface is smooth or hilly, and on whether is is shiny versus reflecting light equally in all directions. We will assume a fixed albedo, a smooth surface, and reflecting equally in all directions.

First we will find how it depends on the phase.

Here is the diagram from section 1, with the addition of a radius at angle α above the radius to the left, which touches the circle at y above the radius to the left.



The part of the planet that is illuminated is not all equally illuminated. It is illuminated the brightest where the sun is directly overhead, and the least where the sun is rising or setting. The level at any point is the former times the cosine of the sun's altitude. You can see this by considering how spread out the light is when it comes in at an angle.

To find the total brightness, we integrate with y going from one edge of the illuminated part visible from earth, to the opposite edge. Note that α can be positive or negative.

$$b = \int_{-r}^{r\cos\theta} c\cos(\theta+\alpha)dy = c\int_{-r}^{r\cos\theta} \cos(\theta + \operatorname{Arcsin}\frac{y}{r})dy$$

$$= c\int_{-r}^{r\cos\theta}\left[\cos\theta\cos(\operatorname{Arcsin}\frac{y}{r}) - \sin\theta\sin(\operatorname{Arcsin}\frac{y}{r})\right]dy$$

$$= c\int_{-r}^{r\cos\theta}\left[\cos\theta\sqrt{1-y^2/r^2} - (\sin\theta)\left(\frac{y}{r}\right)\right]dy$$

$$= \frac{c}{r}\cos\theta\int_{-r}^{r\cos\theta}\sqrt{r^2-y^2}dy \;-\; \frac{c}{r}\sin\theta\int_{-r}^{r\cos\theta} ydy$$

$$= \frac{c}{r}\cos\theta\left[\frac{y}{2}\sqrt{r^2-y^2} + \frac{r^2}{2}\operatorname{Arcsin}\frac{y}{r}\right]\Bigg]_{-r}^{r\cos\theta} \;-\; \frac{c}{r}\sin\theta\left[\frac{y^2}{2}\right]\Bigg]_{-r}^{r\cos\theta}$$

$$= \left\{ \frac{c}{r} \cos\theta \left[\left(\frac{r\cos\theta}{2} \sqrt{r^2 - r^2\cos^2\theta} + \frac{r^2}{2} \operatorname{Arcsin} \frac{r\cos\theta}{r} \right) - \right.\right.$$

$$\left.\left. \left(\frac{-r}{2} \sqrt{r^2 - (-r)^2} + \frac{r^2}{2} \operatorname{Arcsin} \frac{-r}{r} \right) \right] \right\} - \left\{ \frac{c}{r} \sin\theta \left[\frac{r^2\cos^2\theta}{2} - \frac{(-r)^2}{2} \right] \right\}$$

$$= \left\{ \frac{c}{r} \cos\theta \left[\left(\frac{r^2\cos\theta}{2} \sin\theta + \frac{r^2}{2} \left(\frac{\pi}{2} - \theta \right) - \left(0 + \frac{-\pi r^2}{4} \right) \right) \right] \right\} - \left\{ \frac{c}{r} \sin\theta \left[\frac{r^2}{2} (\cos^2\theta - 1) \right] \right\}$$

$$= \left\{ \frac{c}{r} \cos\theta \left[\frac{r^2}{2} \sin\theta\cos\theta + \frac{\pi r^2}{4} - \frac{r^2}{2}\theta + \frac{\pi r^2}{4} \right] \right\} - \left\{ -\frac{cr}{2} \sin^3\theta \right\}$$

$$= \frac{cr}{2} \left[\cos\theta (\sin\theta\cos\theta - \theta + \pi) + \sin^3\theta \right]$$

So, for example, when $\theta = 0$, $b = cr\pi/2$. When $\theta = \pi/2$, $b = cr/2$. Thus, when viewed from the same distance, the full phase is not twice as bright as the half phase; it is π times as bright.

In section 2 we saw that the distance from the planet to earth was

$$h = \sqrt{a^2 + 2ax + p^2}$$

So, taking the distance into account, using the inverse squared law, we find that the brightness of the planet as seen from earth is:

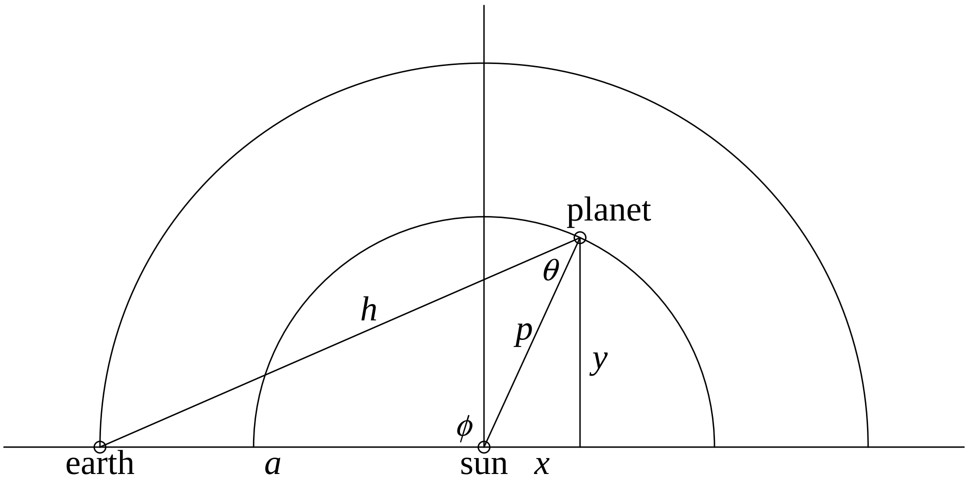
$$B = \frac{b}{h^2} = \frac{\frac{cr}{2} \left[\cos\theta (\sin\theta\cos\theta - \theta + \pi) + \sin^3\theta \right]}{a^2 + 2ax + p^2}$$

To get this purely in terms of θ, you can take the equation at the end of section 2, which is θ in terms of x, and solve it for x in terms of θ, and plug that in for x here. But it is ugly.

6. Find θ based on orbital positions of an inferior planet

In this section we deal with inferior planets.

In the following diagram, we use the same variable names as for the similar diagram in section 2.



The same calculations apply as were done in section 2, up to the equation:

$$\sin\theta = a\sqrt{\frac{1 - x^2/p^2}{a^2 + 2ax + p^2}}$$

But since $0 \le \theta \le \pi$, we cannot use "Arcsin": For each x value from $-p$ to p, there are two possible values of θ.

So, instead, do it this way: By the law of cosines,

$$a^2 = p^2 + h^2 - 2ph\cos\theta = p^2 + a^2 + 2ax + p^2 - 2p\sqrt{a^2 + 2ax + p^2}\cos\theta$$

$$-2ax - 2p^2 = -2p\sqrt{a^2 + 2ax + p^2}\cos\theta$$

$$ax + p^2 = p\sqrt{a^2 + 2ax + p^2}\cos\theta$$

$$\cos\theta = \frac{ax + p^2}{p\sqrt{a^2 + 2ax + p^2}}$$

θ is between 0 and π, which is the interval covered by Arccos, so we can say:

$$\theta = \text{Arccos}\left[\frac{ax + p^2}{p\sqrt{a^2 + 2ax + p^2}}\right]$$

As for brightness, the same formula applies as for superior planets. It would be nice to find the place of maximum brightness by taking dB/dx, setting that to zero, and solving for x, but it is probably too complicated to be solveable.