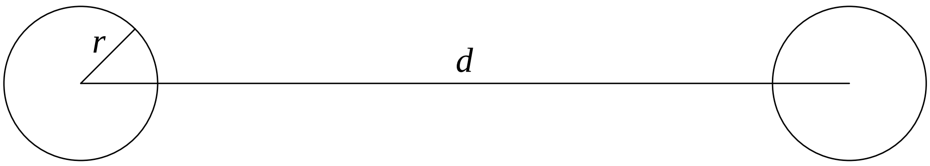
If there were two identical balls positioned motionless in space, far from any other objects, how long would it take for them to fall towards each other and collide, due to their gravity?

1. Deriving the formula

Set the following values:
Radius of the balls: r
Mass of each ball: m
Starting distance between their centers: d
Graviational constant: G



The force on the right object when its distance from the midpoint is s is

$$F = \frac{-Gm^2}{(2s)^2}$$

since the distance between the objects' centers is s. The kinetic energy of the right object after falling to distance s from the midpoint equals the work done on it by gravity. That is the force times the distance x along the way:

$$K = \int_{d/2}^{s} F\ dx = \int_{d/2}^{s} \frac{-Gm^2}{(2x)^2}\ dx = \frac{-Gm^2}{4}\left[\int_{d/2}^{s} \frac{dx}{x^2}\right] = \frac{-Gm^2}{4}\left[\left[\frac{-1}{x}\right]_{d/2}^{s}\right]$$

$$= \frac{-Gm^2}{4}\left[\frac{-1}{s} - \frac{-1}{d/2}\right] = \frac{Gm^2}{4}\left(\frac{1}{s} - \frac{2}{d}\right)$$

But also, $K = mv^2/2$, where v is the velocity. So,

$$\frac{mv^2}{2} = \frac{Gm^2}{4}\left(\frac{1}{s} - \frac{2}{d}\right)$$

$$v = -\sqrt{\frac{Gm}{2}\left(\frac{1}{x} - \frac{2}{d}\right)}$$

where we use the negative square root because it is moving towards the left.

$v = dx/dt$, so $dt = dx/v$, and we can find the time t for the right object to fall from its starting point to where it touches the left object as follows:

$$t = \int_{d/2}^{r} \frac{dx}{-\sqrt{\frac{Gm}{2}\left(\frac{1}{x}-\frac{2}{d}\right)}} = -\sqrt{\frac{2}{Gm}} \int_{d/2}^{r} \frac{dx}{\sqrt{1/x-2/d}} = \sqrt{\frac{2}{Gm}} \int_{r}^{d/2} \frac{dx}{\sqrt{1/x-2/d}}$$

$$= \sqrt{\frac{2}{Gm}} \int_{r}^{d/2} \frac{x\ dx}{\sqrt{x-2x^2/d}} = \sqrt{\frac{2}{Gm}} \left[\left[\frac{\sqrt{x-2x^2/d}}{-2/d}\right]_r^{d/2} - \left(\frac{-d}{4}\right) \int_r^{d/2} \frac{dx}{\sqrt{x-2x^2/d}} \right]$$

$$= \sqrt{\frac{2}{Gm}} \left[\frac{\sqrt{d/2-2(d/2)^2/d}}{-2/d} - \frac{\sqrt{r-2r^2/d}}{-2/d} + \frac{d}{4}\left\{ \left[\frac{-1}{\sqrt{2/d}} \operatorname{Arcsin}\left(\frac{-4x/d+1}{\sqrt{0-(-1)}}\right)\right]_r^{d/2} \right\} \right]$$

$$= \sqrt{\frac{2}{Gm}} \left[\frac{d}{2}\sqrt{r-\frac{2r^2}{d}} - \frac{d\sqrt{d}}{4\sqrt{2}} \left\{ \operatorname{Arcsin}(-2+1) - \operatorname{Arcsin}\left(-\frac{4r}{d}+1\right) \right\} \right]$$

$$= \sqrt{\frac{2}{Gm}} \left[\frac{d}{2}\sqrt{r-\frac{2r^2}{d}} - \frac{d\sqrt{d}}{4\sqrt{2}} \left\{ -\frac{\pi}{2} - \operatorname{Arcsin}\left(1-\frac{4r}{d}\right) \right\} \right]$$

$$= \sqrt{\frac{2}{Gm}} \left[\frac{d}{2}\sqrt{r-\frac{2r^2}{d}} + \frac{d\sqrt{d}}{4\sqrt{2}} \left\{ \frac{\pi}{2} + \operatorname{Arcsin}\left(1-\frac{4r}{d}\right) \right\} \right]$$

If $r = 0$, point masses, this simplifies to

$$t = \sqrt{\frac{2}{Gm}} \frac{d\sqrt{d}}{4\sqrt{2}} \pi = \frac{\pi d}{4} \sqrt{\frac{d}{Gm}}$$

2. Applying the formula

Assume the balls are positioned one meter apart ($d = 1$ m), have a mass of 1 kilogram ($m = 1$ kg), and are made of iron. The density of iron is 7870 kilograms per cubic meter. So the volume V is

$$V = \frac{1}{7870}\,\mathrm{m}^3 = \frac{4}{3}\pi r^3$$

and $r = 0.031187$ m. The gravitational constant $G = 6.67384 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$. Plugging these values into the formula, we find the time for the fall is 95,492 seconds: 26 hours, 31 minutes, and 32 seconds. The speed of each when they collide is 3.167365×10^{-5} m/s: 32 seconds to move one millimeter.

If we set $r = 0$, point masses, we find the time is 96,140 seconds: 26 hours, 42 minutes, and 20 seconds. The speed when they meet would be infinite, according to Newtonian physics.

If we keep the original balls, but start 1 kilometer apart, the time works out to about 3,040,198,000 seconds, which is 96 years.

If we start with two neutrons as the masses, one meter apart, we have $m = 1.674927 \times 10^{-27}$ kg and $d = 1$ m. Consider them to be point masses, which for this purpose is practically true. The time works out to 2.349114×10^{18} seconds. That is about 74 billion years. Of course actually they would almost certainly decay before then, their half life being about 15 minutes.

Returning to the iron balls one meter apart, each with a mass of one kilogram, if you added some extra electrons to each ball, how many would be needed for the electrical repulsion to counteract the gravitational attraction? We want

$$\frac{Gm^2}{d^2} = \frac{kq^2}{d^2}$$

where k is the Coulomb constant, $8.987552 \times 10^9 \mathrm{Nm}^2\mathrm{C}^{-2}$, and q is the charge in coulombs. Note that the distance d cancels out, so the answer is the same regardless of the distance. We see that

$$q = m\sqrt{\frac{G}{k}}$$

So $q = 8.61722 \times 10^{-11}$ coulombs. That is about 537,845,000 electrons. Actually, because of the repulsion, the electrons would distribute themselves unequally, more of them being on the far side of the balls, which would reduce the repulsion slightly; so you would need to increase the number of electrons a little.